Worksheet on Linearization

Example 1. Linearization with one variable

Linearize the following equation around $\bar{x}$ =3:

$$f(x) = 3x^3 + 5x^2 + 27$$

(i) Write the Taylor's series expansion:

f(x) = $f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x})$

(ii) Evaluate $f(\bar{x})$ = $3(3)^3 + 5(3)^2 + 27 = 3(27) + 5(9) + 27 = 153$

(iii) What is the derivative of the function? $\left(\frac{df}{dx}\right)$ = $9x^2 + 10x$

(iv) Evaluate $f'(\bar{x})$ = $9(3)^2 + 10(3) = 111$

(v) Write the final linear expression *f(x)* = $153 + 111(x-3)$

Example 2. Linearization with two variables

Linearize the following equation around $\bar{x} = 2$ and $\bar{y} = 2$:

$$f(x, y) = 3xy + y^2 - 3x^2$$

(i) Write the Taylor's series expansion:

f(x,y) = $f(\bar{x},\bar{y}) + \left.\frac{df}{dx}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} (x-\bar{x}) + \left.\frac{df}{dy}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} (y-\bar{y})$

(ii) Evaluate $f(\bar{x}, \bar{y})$ = $3(2)(2) + (2)^2 - 3(2)^2 = 4$

(iii) What are the partial derivatives of the function?

$\frac{\partial f}{\partial x}$ = $3y - 6x$

$\frac{\partial f}{\partial y}$ = $3x + 2y$

(iv) Evaluate $\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}}$ = $6 - 12 = -6$

$\left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}}$ = $6 + 4 = 10$

(v) Write the final linear expression:

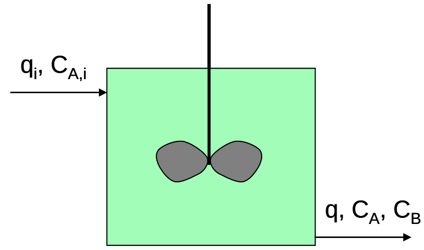
f(x) = $4 - 6(x-2) + 10(y-2)$

Example 3. CSTR with three variables

Species Balance

$$\frac{\partial n_A}{\partial t} = \dot{n}_{A,in} - \dot{n}_{A,out} + r_A V$$

$$n_A = C_A V$$



$$-r_A = k_1 C_A^2 - k_2 C_A C_B$$

(i) Write the transient mole balance for species A:

$$V\frac{dC_A}{dt} = C_{A,in}\, q_{in} - C_A\, q + \left(-k_1 C_A^2 + k_2 C_A C_B\right) V$$

(ii) Assume k_1, k_2, q_i, q, and V are constant. The function to linearize is just the RHS of the above equation! The variables are $C_{A,in}$, C_A, and C_B.

(iii) At steady state, what is the value of $V\frac{dC_A}{dt}$? 0. Time derivatives are zero

Therefore, $f\left(\bar{C}_{A,i}, \bar{C}_A, \bar{C}_B\right) = 0$.

(iv) Write the Taylor's series expansion:

$$f\left(C_{A,i}, C_A, C_B\right) = f\left(\bar{C}_{A,in}, \bar{C}_A, \bar{C}_B\right) + \left.\frac{df}{dC_{A,in}}\right|_{\substack{C_{A,in}=\bar{C}_{A,in} \\ C_A=\bar{C}_A \\ C_B=\bar{C}_B}} \left(C_{A,in} - \bar{C}_{A,in}\right) + \left.\frac{df}{dC_A}\right|_{ss}\left(C_A - \bar{C}_A\right) + \frac{df}{dC_B}\left(C_B - \bar{C}_B\right)$$

(v) What are the partial derivatives of the function?

$$\frac{\partial f}{\partial C_{A,i}} = q_{in}$$

$$\frac{\partial f}{\partial C_A} = -q - 2k_1 C_A V + k_2 C_B V$$

$$\frac{\partial f}{\partial C_B} = k_2 C_A V$$

(vi) Evaluate

$$\left.\frac{\partial f}{\partial C_{A,i}}\right|_{ss} = q_{in} = \alpha_1$$

$$\left.\frac{\partial f}{\partial C_A}\right|_{ss} = -q - 2k_1 \bar{C}_A V + k_2 \bar{C}_B V = \alpha_2$$

$$\left.\frac{\partial f}{\partial C_B}\right|_{ss} = k_2 \bar{C}_A V = \alpha_3$$

(vii) Write the final linear expression:

$$f\left(C_{A,i}, C_A, C_B\right) = 0 + \alpha_1\left(C_{A,i} - \bar{C}_{A,i}\right) + \alpha_2\left(C_A - \bar{C}_A\right) + \alpha_3\left(C_B - \bar{C}_B\right)$$

constants

(viii) Now introduce deviation variables (the prime here is <u>not</u> a derivative):

$$C'_{A,i} = C_{A,i} - \overline{C}_{A,i}$$

$$C'_A = C_A - \overline{C}_A$$

$$C'_B = C_B - \overline{C}_B$$

(ix) The transient linearized equation now becomes:

$$V\frac{dC_A}{dt} = V\frac{dC'_A}{dt} = \underline{\alpha_1}\, C'_{A,i} + \underline{\quad \alpha_2 \quad}\, C'_A + \underline{\quad \alpha_3 \quad}\, C'_B$$

(x) All of the underlined terms above are constants, since they were evaluated at the steady-state condition. For convenience in this equation, call the second constant c_1. The standard form for solving this equation using Laplace transforms is:

$$\tau \frac{dy'}{dt} + y' = f(x_1, x_2, x_3, etc.)$$

$$V\frac{dC_A}{dt} = \alpha_1 C'_{A,i} + \alpha_2 C'_A + \alpha_3 C'_B$$

Put the equation in (ix) into standard form:

$$\frac{V}{-\alpha_2\,\cancel{c_1}}\frac{dC'_A}{dt} + C'_A = \underline{\left(\frac{\alpha_1}{-\alpha_2}\right)}\, C'_{A,i} + \underline{\left(\frac{\alpha_3}{-\alpha_2}\right)}\, C'_B$$

$$\frac{dC'_A}{dt} = \frac{d\left(C_A - \overline{C}_A\right)}{dt} = \frac{dC_A}{dt} - \cancel{\frac{d\overline{C}_A}{dt}} \rightarrow 0$$

because it is a constant

$$\frac{dC'_A}{dt} = \frac{dC_A}{dt}$$