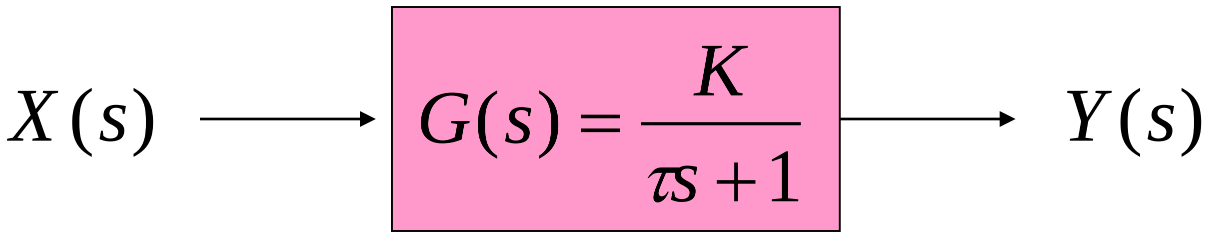
Response of First Order Systems



First Order Functions

- Time Domain

$$\tau \frac{dy}{dt} + y = Kx$$

- Laplace Domain

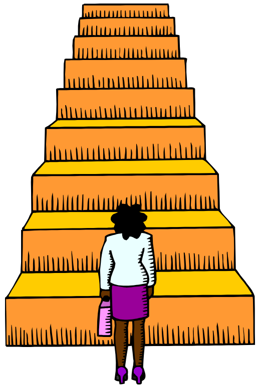
$$\frac{Y(s)}{X(s)} = G(s) = \frac{K}{\tau s + 1}$$

Input Functions (i.e., X(s))

Step	$u(s)=\dfrac{M}{s}$	(5-6)
Ramp	$u(s)=\dfrac{a}{s^2}$	(5-8)
Rectangular pulse	$u(s)=\dfrac{h}{s}\left(1-e^{-t_w s}\right)$	(5-11)
Triangular pulse	$u(s)=\dfrac{2}{t_w}\left(\dfrac{1-2e^{-t_w s/2}+e^{-t_w s}}{s^2}\right)$	(5-13)
Sine wave	$u(s)=\dfrac{A\omega}{s^2+\omega^2}$	(5-15)
Impulse	$u(s)=a$	p. 76

(slope=$t_w/2$)

Response to Step



$$Y(s) = X(s)G(s)$$

$$X(s) = \frac{M}{s}$$

$$G(s) = \frac{K}{\tau s + 1}$$

$$Y(s) = \frac{KM}{s(\tau s + 1)}$$

$$y(t) = KM\left(1 - e^{-t/\tau}\right)$$

(5-18)

Response to Ramp

$$Y(s) = X(s)G(s)$$

$$X(s) = \frac{a}{s^2}$$

$$G(s) = \frac{K}{\tau s + 1}$$

$$Y(s) = \frac{Ka}{s^2(\tau s + 1)}$$

$$y(t) = Ka\tau\left(e^{-t/\tau} - 1\right) + Kat$$

(5-22)

Response to Sine Wave

$$Y(s) = X(s)G(s)$$

$$X(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$G(s) = \frac{K}{\tau s + 1}$$

$$Y(s) = \frac{KA\omega}{\left(s^2 + \omega^2\right)\left(\tau s + 1\right)} = KA\omega\left[\frac{a}{\tau s + 1} + \frac{bs + c}{s^2 + \omega^2}\right]$$

$$y(t) = \frac{Ka}{\omega^2\tau^2 + 1}\left(\omega\tau e^{-t/\tau} - \omega\tau\cos\omega t + \sin\omega t\right) \quad (5\text{-}25)$$

Time Delays (θ)

In time domain:

- Replace t with (t-θ) and multiply by S(t-θ)

$$f\left(t-\theta\right)\cdot S\left(t-\theta\right)$$

In Laplace domain

- Multiply by $e^{-\theta s}$

$$e^{-\theta s}F(s)$$

Example: FOPDT

- This is a response of a first order model to a step function M
- First order with a step function is:

$$Y(s) = \frac{KM}{s(\tau s + 1)}$$

$$y(t) = KM\left(1 - e^{-t/\tau}\right)$$

- Now add time delay

$$Y(s) = \frac{KM \cdot e^{-\theta s}}{s(\tau s + 1)}$$

$$y(t) = KM\left(1 - e^{-(t-\theta)/\tau}\right) \cdot S(t - \theta)$$

Pumped Tank Example

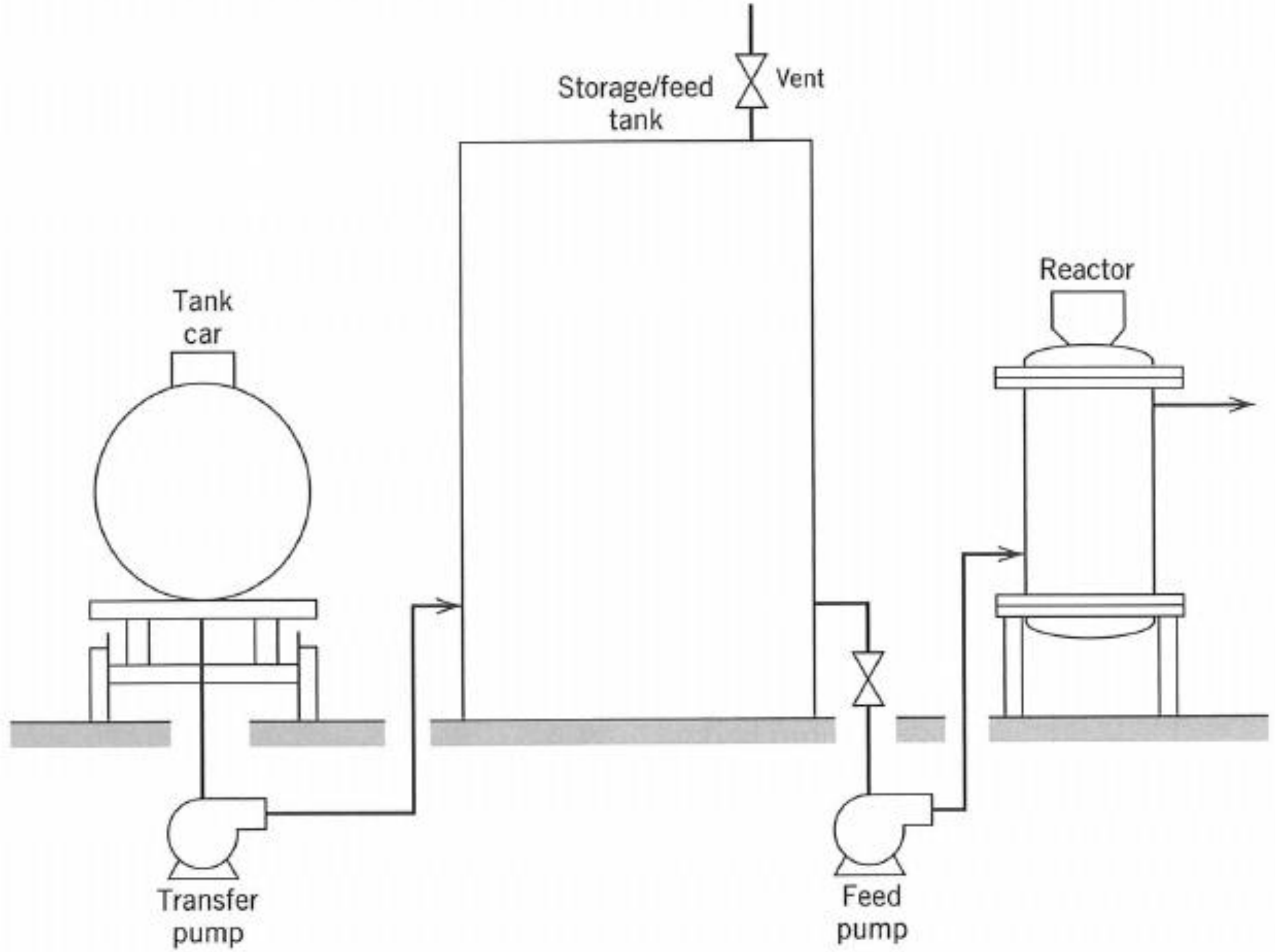


Figure 5.6 Unloading and storage facility for a continuous reactor.

Integrating Process

- Pumped Tank

$$A\frac{dh}{dt} = q_i - q$$

$$sAH'(s) = Q_i'(s) - Q'(s)$$

$$H'(s) = \frac{1}{sA}\left[Q_i'(s) - Q'(s)\right]$$

$$\frac{H'(s)}{Q_i'(s)} = \frac{1}{sA}$$

$$\frac{H'(s)}{Q'(s)} = -\frac{1}{sA}$$

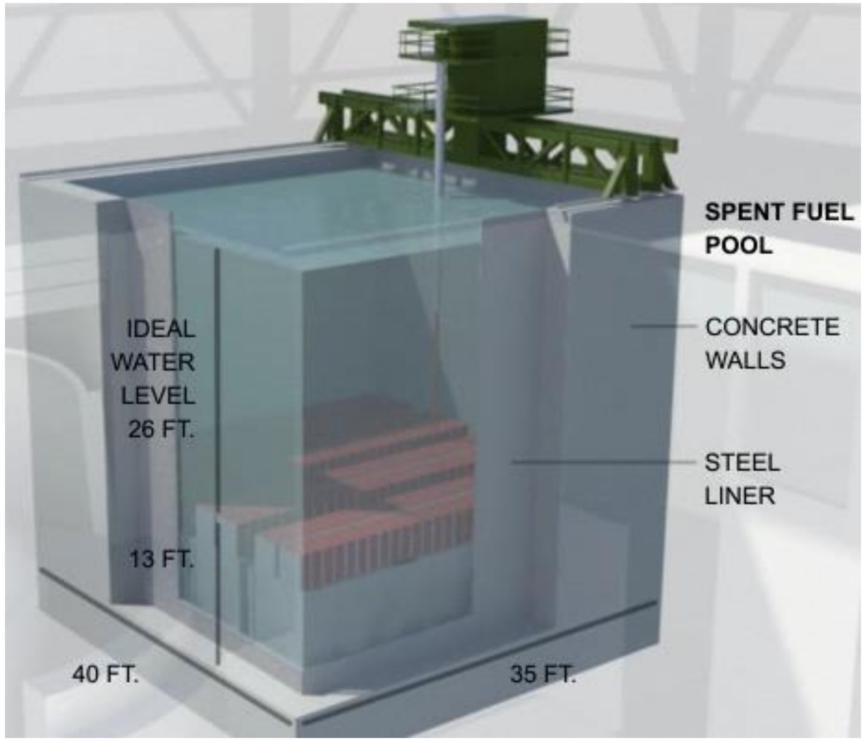
- This is not a first order model
- Called an integrating process (no steady-state gain)
- Step function in q or q_i results in ramp in h

$$Q_i'(s) = \frac{M}{s}$$

$$H'(s) = \frac{M}{s^2 A}$$

Fukushima Application

How long until the water level reaches the spent nuclear fuel rods if the power is shut off? When the spent nuclear fuel was exposed after 3 1/2 days, was there likely a leak caused by the earthquake or was the level loss due only to vaporization?

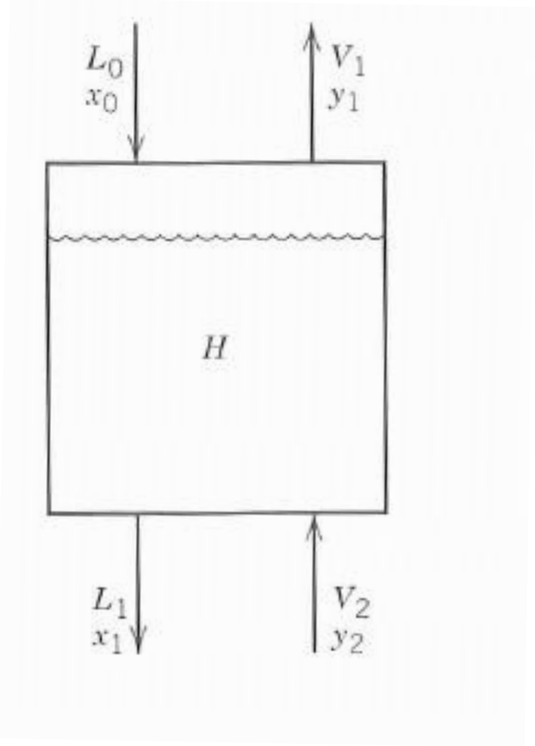


Problem 4.7 – Distillation Stage

H is molar holdup, L and V are molar flow rates
y's and x's are mole fractions in vapor and liquid

- Wanted:

$$\frac{X_1'(s)}{X_0'(s)} \quad \frac{X_1'(s)}{Y_2'(s)} \quad \frac{Y_1'(s)}{X_0'(s)} \quad \frac{Y_1'(s)}{Y_2'(s)}$$



Stirred tank blending system
(or stage on distillation column)

- Given:

$$\frac{dH}{dt} = L_0 + V_2 - (L_1 + V_1)$$

$$\frac{dx_1 H}{dt} = x_0 L_0 + y_2 V_2 - (x_1 L_1 + y_1 V_1)$$

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

Vapor / Liquid composition correlation

Assumptions

- Molar holdup H is constant

$$\frac{dH}{dt} = 0$$

- Constant molal overflow

$$L_0 = L_1$$
$$V_1 = V_2$$

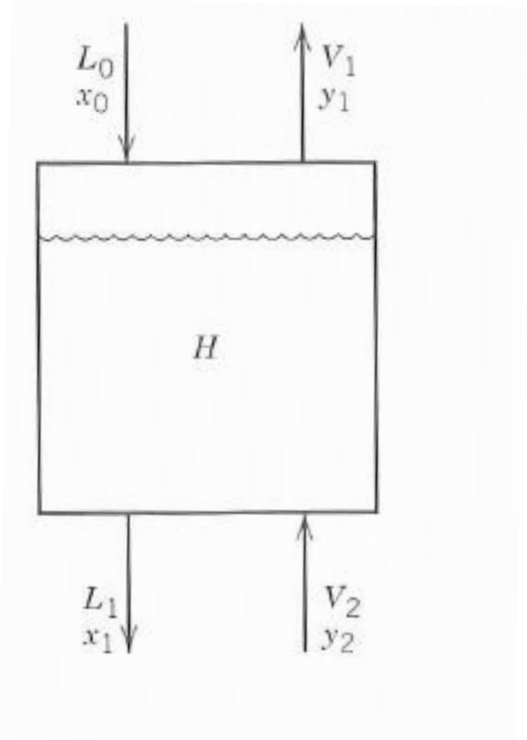
- Simplification: only use *L* and *V* (no subscripts)

$$\frac{dx_1 H}{dt} = x_0 L + y_2 V - \left(x_1 L + y_1 V\right)$$
$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

Distillation Stage

$$\frac{dx_1 H}{dt} = x_0 L + y_2 V - (x_1 L + y_1 V)$$

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$



$\frac{X_1'(s)}{X_0'(s)}$ $\frac{X_1'(s)}{Y_2'(s)}$ $\frac{Y_1'(s)}{X_0'(s)}$ $\frac{Y_1'(s)}{Y_2'(s)}$