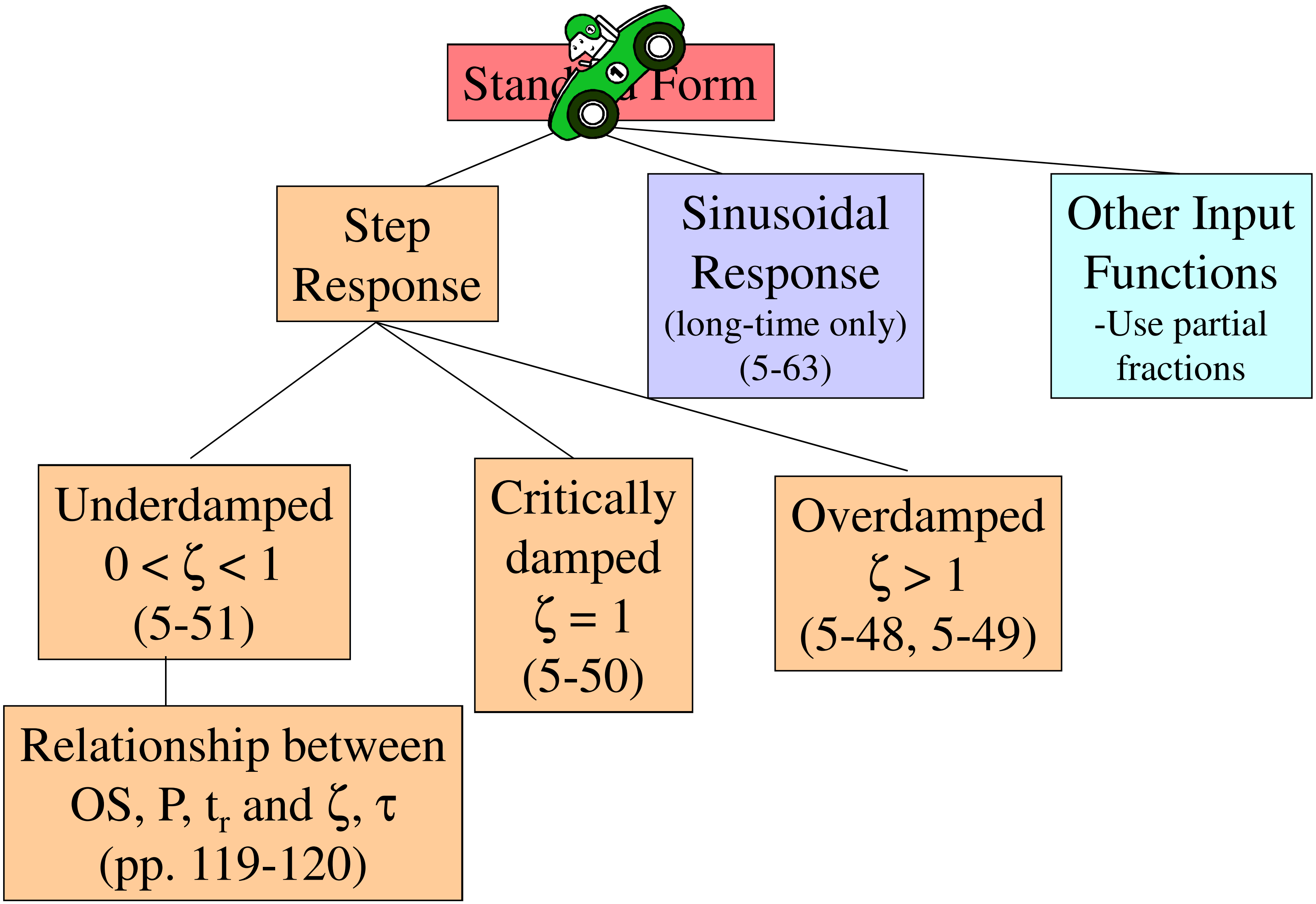
# Class 22

Complex Transfer Functions

# Road Map for 2$^{nd}$ Order Equations



# What About Higher Order Systems?

$$G(s) = \frac{30}{24s^3 + 20s^2 + 10s + 2}$$

or

$$G(s) = \frac{30s^2 + 6s + 7}{24s^3 + 20s^2 + 10s + 2}$$

A polynomial in the numerator is called a lead element

A polynomial in the denominator is called a lag element


Method to describe stability behavior of system using simple analysis of transfer function

# Poles and Zeros

- Transfer function can usually be represented as a ratio of two polynomials in the Laplace variable *s* along with a possible delay term:

$$G(s) = \frac{Z(s)}{P(s)} e^{-\theta s}$$

where $Z(s) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0$

and $P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

Roots of Z(s) = "zeros"

Roots of P(s) = "poles"

# Different Forms of G(s)

$$G(s) = \frac{b_m (s - z_1)(s - z_2)\ldots(s - z_m)}{a_n (s - p_1)(s - p_2)\ldots(s - p_n)} e^{-\theta s}$$

so $z_1$, $z_2$, …, $z_n$ are the zeros
and $p_1$, $p_2$, …, $p_n$ are the poles

Alternatively, in time constant form,

$$G(s) = \frac{b_m (-z_1)(-z_2)\ldots(-z_m)(\tau_{l1} s + 1)(\tau_{l2} s + 1)\ldots(\tau_{lm} s + 1)}{a_n (-p_1)(-p_2)\ldots(-p_n)(\tau_1 s + 1)(\tau_2 s + 1)\ldots(\tau_n s + 1)} e^{-\theta s}$$

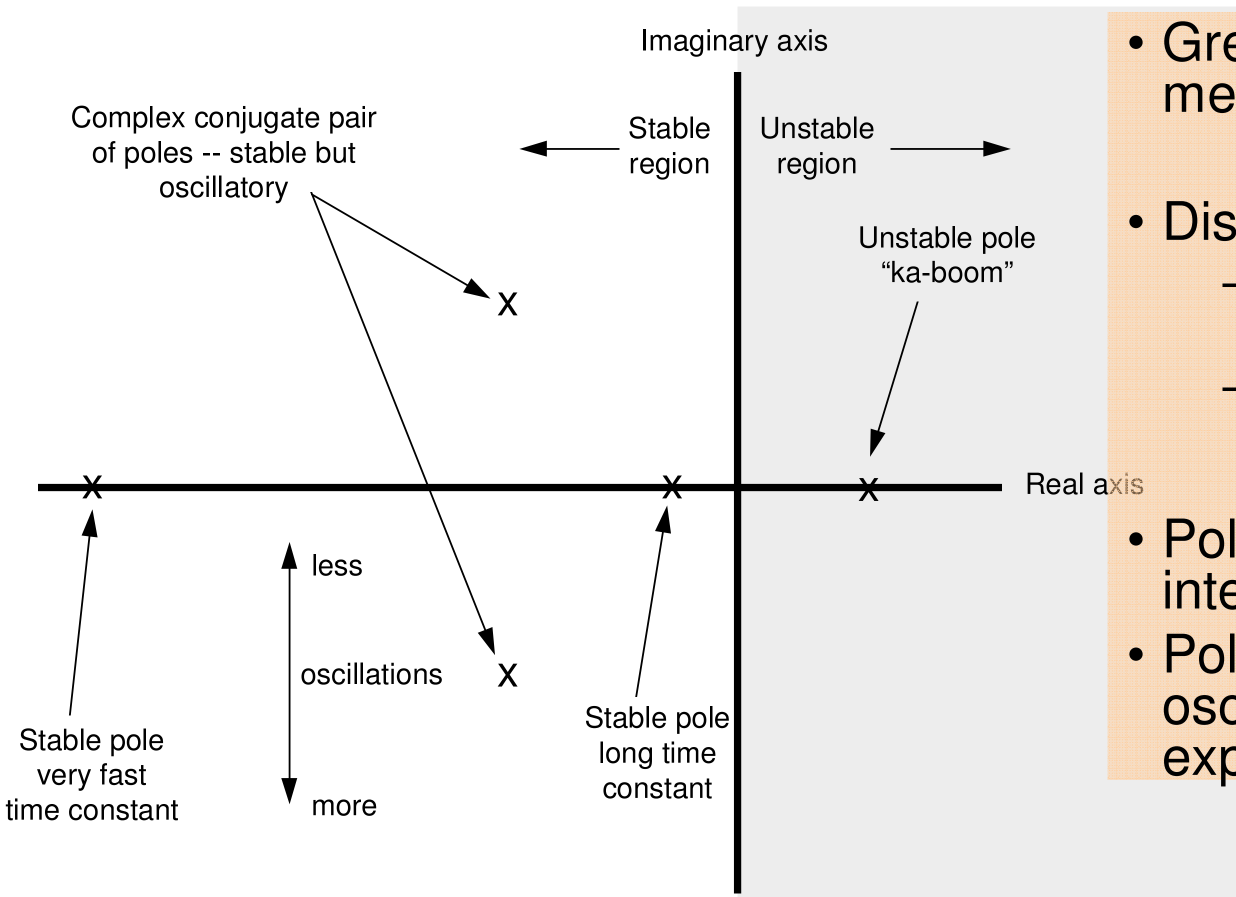
$$= K \frac{(\tau_{l1} s + 1)(\tau_{l2} s + 1)\ldots(\tau_{lm} s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)\ldots(\tau_n s + 1)} e^{-\theta s}$$

$n \geq m$ to be physically realizable

so $-1/\tau_{l1}$, $-1/\tau_{l2}$, …, $-1/\tau_{ln}$ are the zeros
and $-1/\tau_1$, $-1/\tau_2$, …, $-1/\tau_n$ are the poles

# So Who Cares?

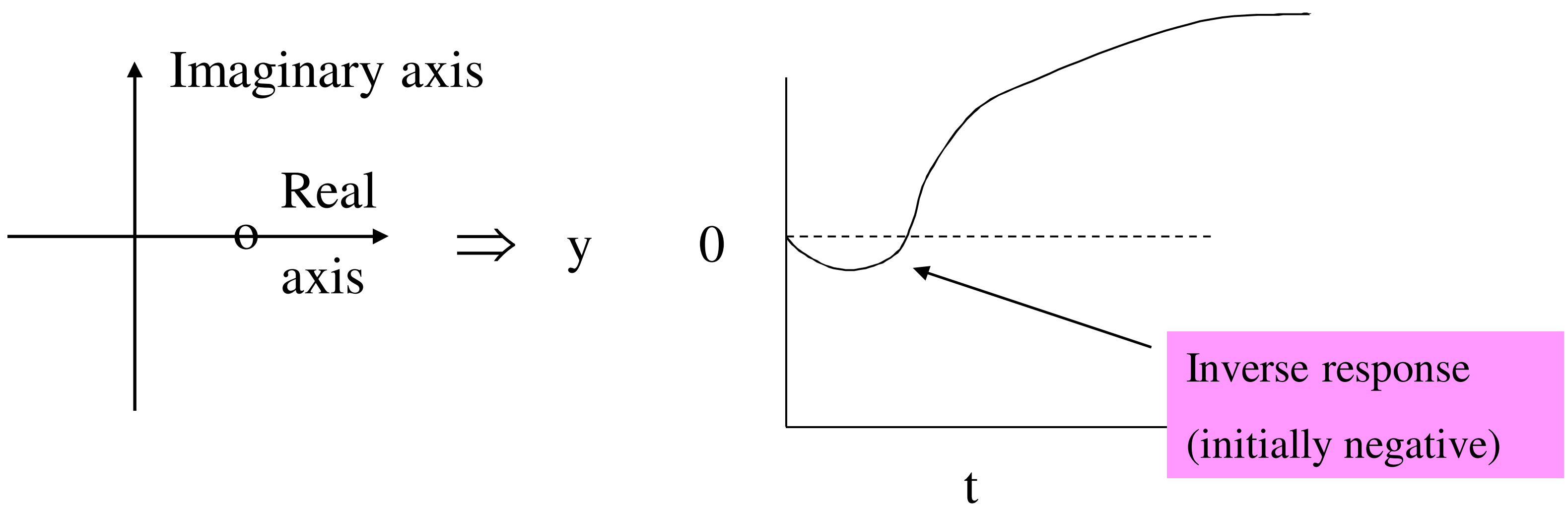
- Poles show the stability of the process
- Zeros show some dynamics (lead-lag)
- Plot poles on real vs imaginary axes with “×”




- Grey area (positive poles) means unstable behavior
- Distance from origin means
  - More oscillations (y direction)
  - Faster response due to shorter time constant (x direction)
- Pole on origin means integrating process
- Poles on y axis mean pure oscillatory behavior with no exponential damping

# What Do Zeros Tell Us?

- Zeros have no effect on system stability.
- *Zero in right half plane:* may result in an inverse response to a step change in the input




- *Zero in left half plane:* may result in "overshoot" during a step response (see Fig. 6.3).

Chapter 6

## *Example 6.2*

For the case of a single zero in an overdamped second-order transfer function,

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad (6\text{-}14)$$

calculate the response to the step input of magnitude *M* and plot the results qualitatively.

**Solution**

The response of this system to a step change in input is

$$y(t) = KM\left(1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}\right) \qquad (6\text{-}15)$$

Note that $y(t \rightarrow \infty) = KM$ as expected; hence, the effect of including the single zero does not change the final value nor does it change the number or location of the response modes. But the zero does affect how the response modes (exponential terms) are weighted in the solution, Eq. 6-15.

A certain amount of mathematical analysis (see Exercises 6.4, 6.5, and 6.6) will show that there are three types of responses involved here:

**Case a:** $\tau_a > \tau_1$

**Case b:** $0 < \tau_a \leq \tau_1$

**Case c:** $\tau_a < 0$

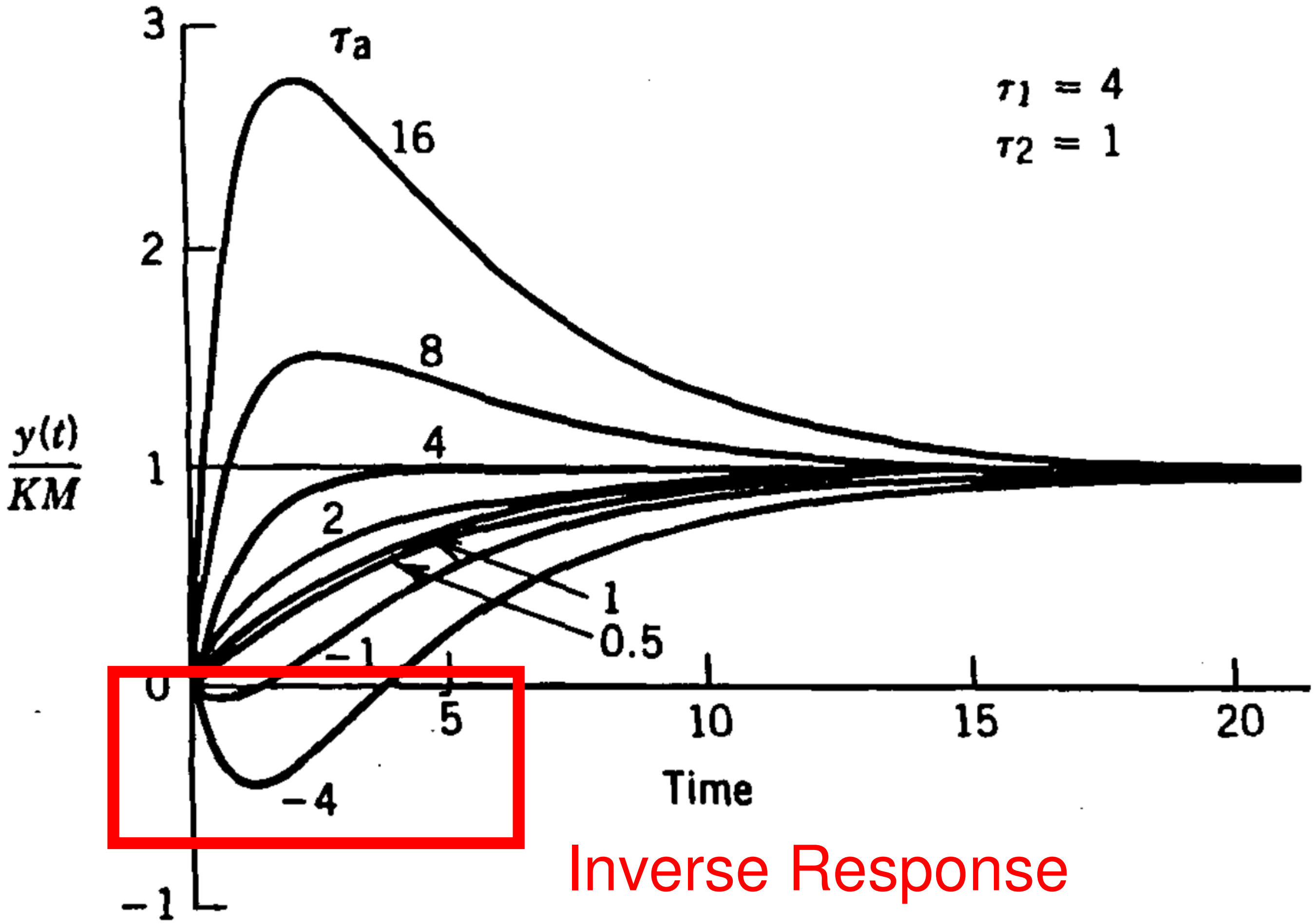



**Figure 6.3.** Step response of an overdamped second-order system (Eq. 6-14) with a single zero.

See page 134 for examples of inverse response

- Increase of steam to reboiler initially causes frothing/spillage on first trays

# Example Problem

$$G(s)=\frac{30}{24s^3+20s^2+10s+2}\Rightarrow\frac{15}{12s^3+10s^2+5s+1}\Rightarrow\frac{15}{(3s+1)(4s^2+2s+1)}$$

Put in pole-zero format:

$$G(s)=\frac{15}{3\left(s+\frac{1}{3}\right)4\left(s^2+\frac{1}{2}s+\frac{1}{4}\right)}$$

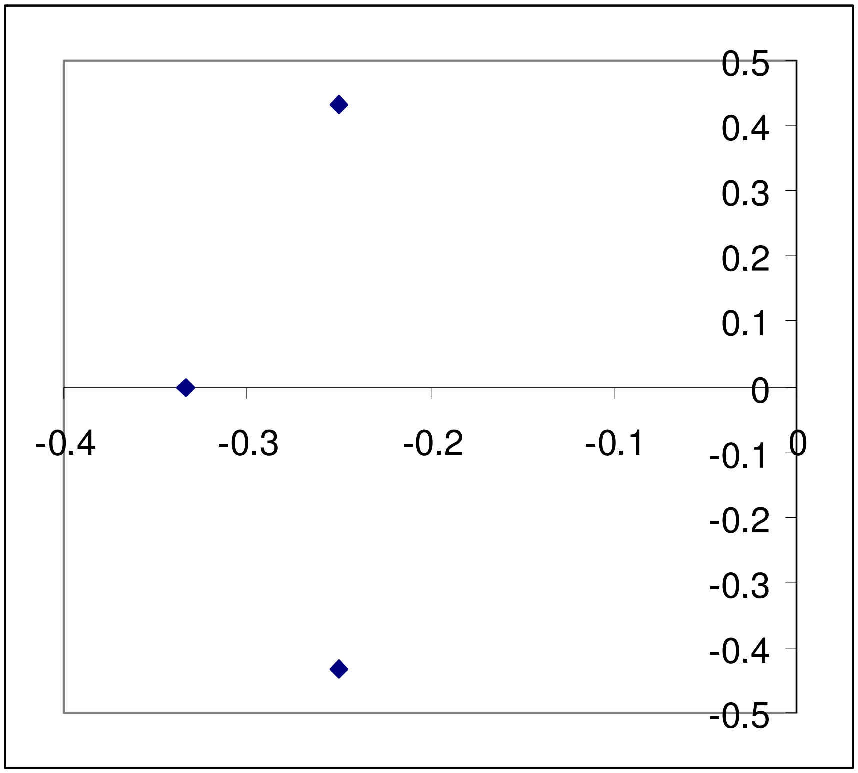
Convert to sine-cosine form:

$$G(s)=\frac{15}{3\left(s+\frac{1}{3}\right)4\left(s^2+\frac{1}{2}s+\frac{1}{4}\right)}=\frac{15}{12\left(s+\frac{1}{3}\right)\left[\left(s+\frac{1}{4}\right)^2+\frac{3}{16}\right]}$$

Find poles:

(-1/3,0) $\left(-\frac{1}{4},\frac{\sqrt{3}}{4}j\right)$ $\left(-\frac{1}{4},-\frac{\sqrt{3}}{4}j\right)$

# Now Plot



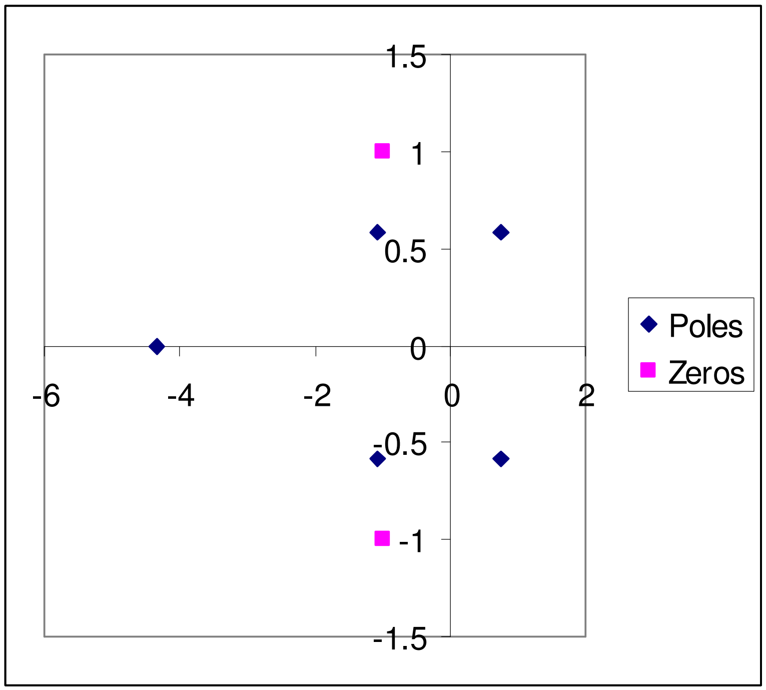

- All left-hand half
  - Exponential decay (good!)
- One imaginary conjugate pair
  - Oscillatory behavior

# Example: Problem 6.1

$$G(s) = \frac{0.7\left(s^2 + 2s + 2\right)}{\left(s^5 + 5s^4 + 2s^3 - 4s^2 + 6\right)}$$

Find zeros and poles (use Mathcad)

| | |
|---|---|
| **Poles** | |
| -4.345 | 0.000 |
| 0.756 | 0.583 |
| -1.083 | 0.585 |
| -1.083 | -0.585 |
| 0.756 | -0.583 |
| **Zeros** | |
| -1 | 1 |
| -1 | -1 |




- Two poles in unstable area
- Any input or disturbance action will cause growth beyond bounds

# Polynomial Approximations to $e^{-\theta s}$

Wanted: polynomial approximations to $e^{-\theta s}$:

Why: Analysis of transfer functions

Two widely used approximations are:

1. ***Taylor Series Expansion:***

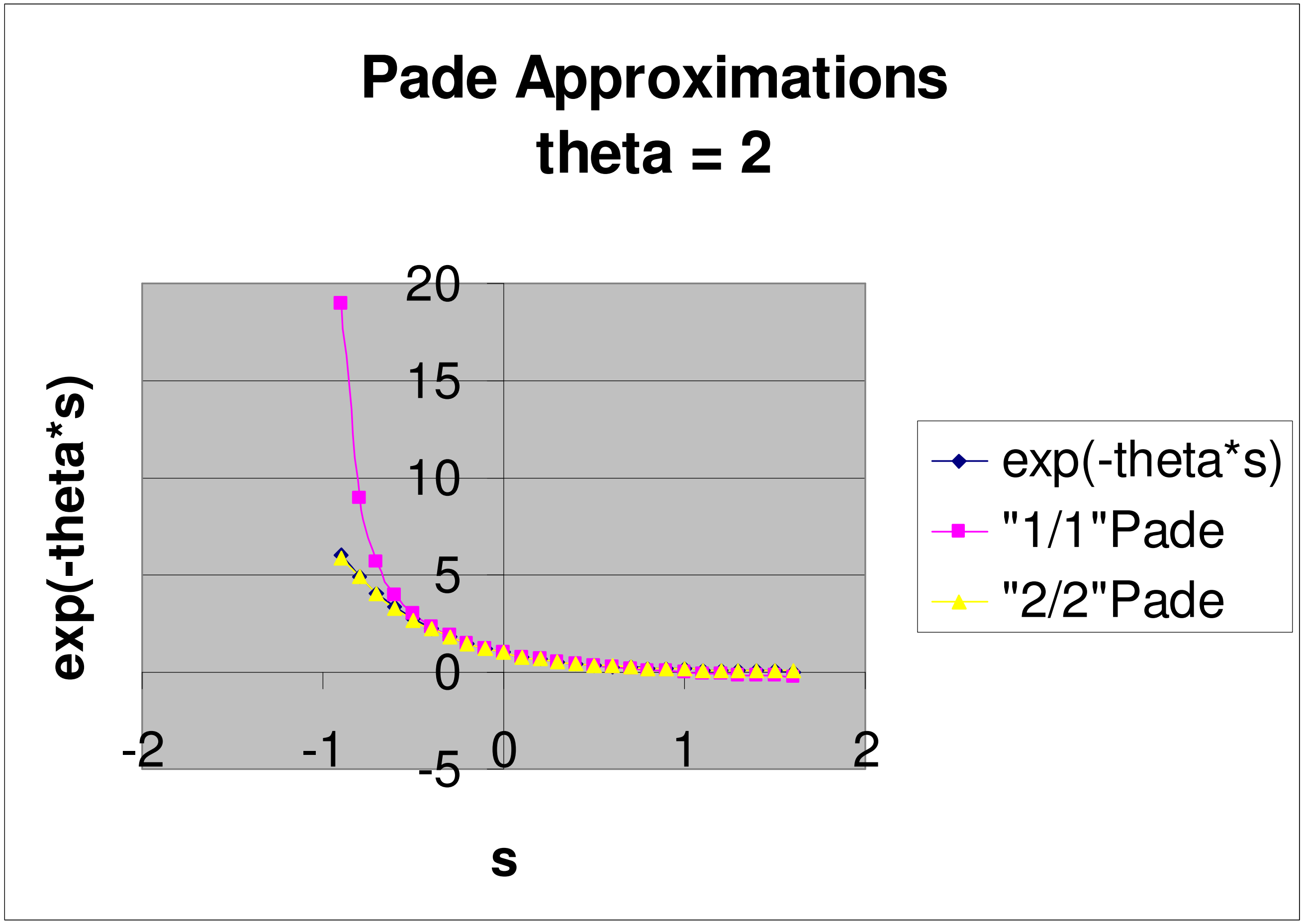
$$e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2!} - \frac{\theta^3 s^3}{3!} + \frac{\theta^4 s^4}{4!} + \ldots \qquad (6\text{-}34)$$

The approximation is obtained by truncating after only a few terms.

2. ***Padé Approximations:***

Many approximations are available. For example, the 1/1 approximation is,

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2} s}{1 + \frac{\theta}{2} s} \qquad (6\text{-}35)$$




***Implications for Control***:

- Time delays can be a challenge for control because they involve a delay of information
- Pade approximation often used when $e^{-\theta s}$ is in denominator

# Taylor Approximation of Higher-Order Transfer Functions

Goal: Approximate high-order transfer function models with lower-order models that have similar dynamic and steady-state characteristics.

- For small values of $s$,

$$e^{-\theta_0 s} \approx 1 - \theta_0 s \qquad (6\text{-}57)$$

(use for numerator terms)

- An alternative first-order approximation consists of the transfer function,

$$e^{-\theta_0 s} = \frac{1}{e^{\theta_0 s}} \approx \frac{1}{1 + \theta_0 s} \qquad (6\text{-}58)$$

(use for denominator terms for non-dominant time constants)

## *Example 6.4*

Consider a transfer function:

$$G(s) = \frac{K(-0.1s+1)}{(5s+1)(3s+1)(0.5s+1)} \qquad (6\text{-}59)$$

Derive an approximate first-order-plus-time-delay model,

$$\tilde{G}(s) = \frac{Ke^{-\theta s}}{\tau s+1} \qquad (6\text{-}60)$$

using the Taylor series expansions of Eqs. 6-57 and 6-58.

**Solution**

(a) The dominant time constant ($\tau = 5$) is retained. Applying the approximations in (6-57) and (6-58) gives:

$$-0.1s + 1 \approx e^{-0.1s} \qquad (6\text{-}61)$$

(numerator)

and

$$\frac{1}{3s+1} \approx e^{-3s} \qquad \frac{1}{0.5s+1} \approx e^{-0.5s} \qquad (6\text{-}62)$$

(denominator terms)

Substitution into (6-59) gives the Taylor series approximation, $\tilde{G}_{TS}(s)$:

$$\tilde{G}_{TS}(s) = \frac{Ke^{-0.1s}e^{-3s}e^{-0.5s}}{5s+1} = \frac{Ke^{-3.6s}}{5s+1} \qquad (6\text{-}63)$$

and *G(s)* can be approximated as:

$$\tilde{G}_{Sk}(s) = \frac{Ke^{-2.1s}}{6.5s+1} \qquad (6\text{-}64)$$

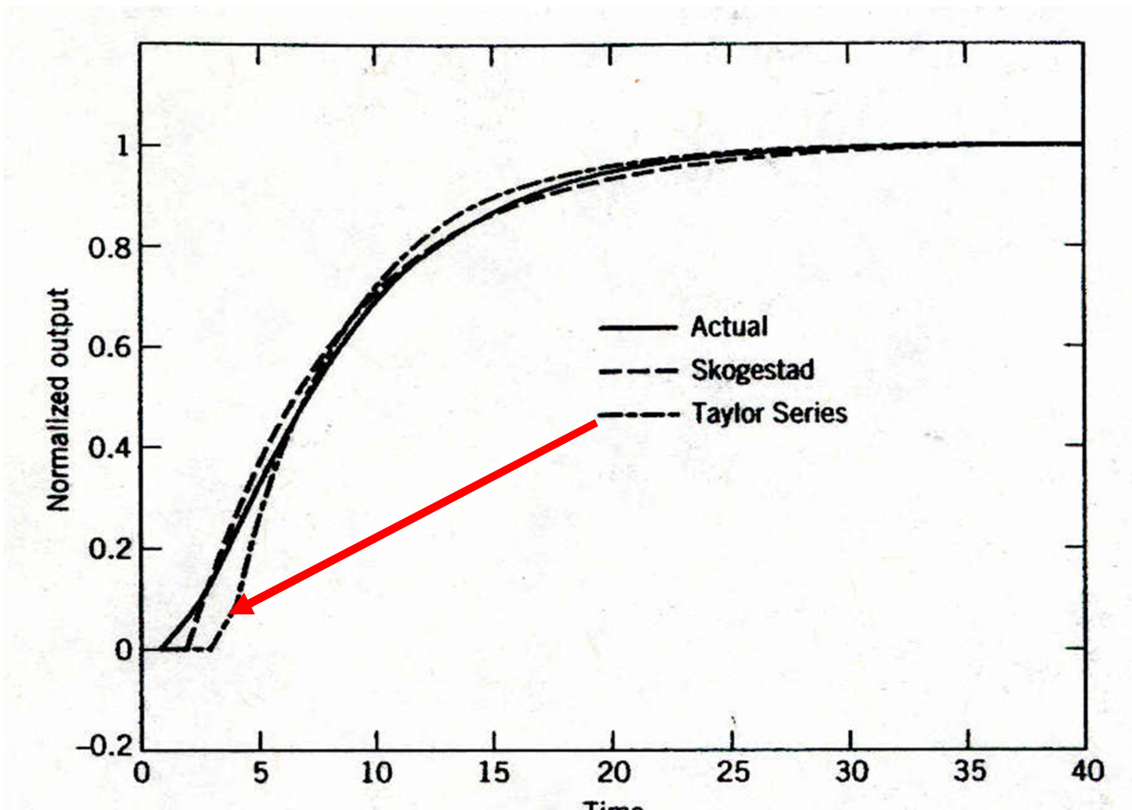
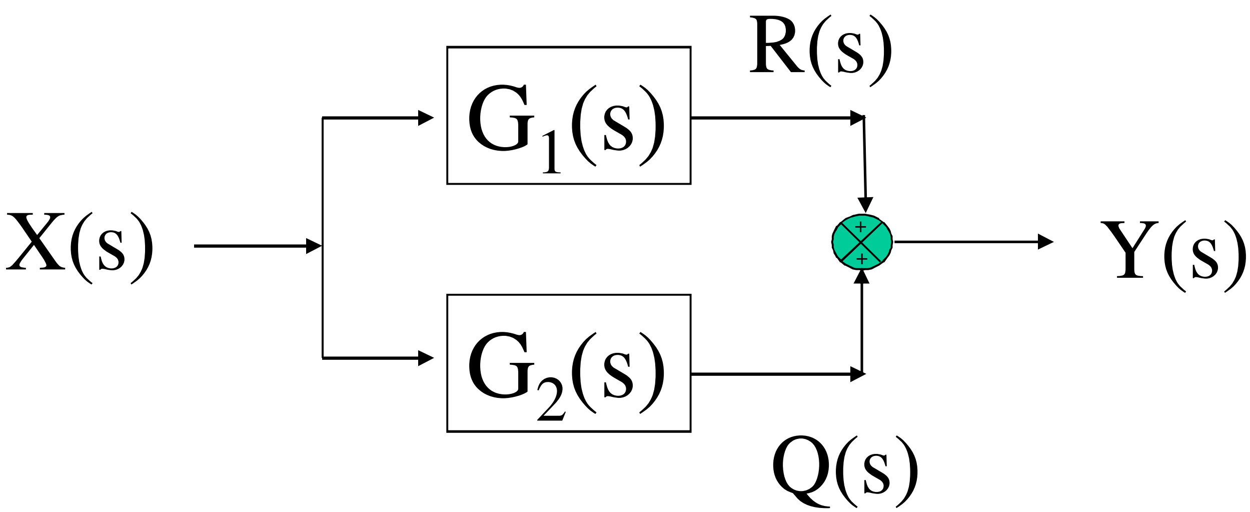



Figure 6.10 Comparison of the actual and approximate models for Example 6.4.

Skogestad's method provides better agreement with the actual response.

# Example: Parallel Processes




$G_1(s)$ is 1st order
$G_2(s)$ is 2nd order

# Parallel Process (cont.)

$$\frac{Y(s)}{X(s)} = G_{overall}(s) = G_1(s) + G_2(s)$$

$$\frac{Y(s)}{X(s)} = \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2^2 s^2 + 2\zeta\tau_2 s + 1}$$

$$= \frac{K_1\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right) + K_2\left(\tau_1 s + 1\right)}{\left(\tau_1 s + 1\right)\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right)}$$

$$= \frac{K_1\tau_2^2 s^2 + \left(K_2 + 2\zeta\tau_2\right)s + K_1 + K_2}{\left(\tau_1 s + 1\right)\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right)}$$

Now put in standard form:

$$= \frac{K'\left(as^2 + bs + 1\right)}{\left(\tau_1 s + 1\right)\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right)}$$

**Moral:**
2 systems in parallel
give lead-lag <u>and</u>
complicated pole-zero form

# Homework Hint on Prob 6.7

See online hint, because I changed the problem a little bit!