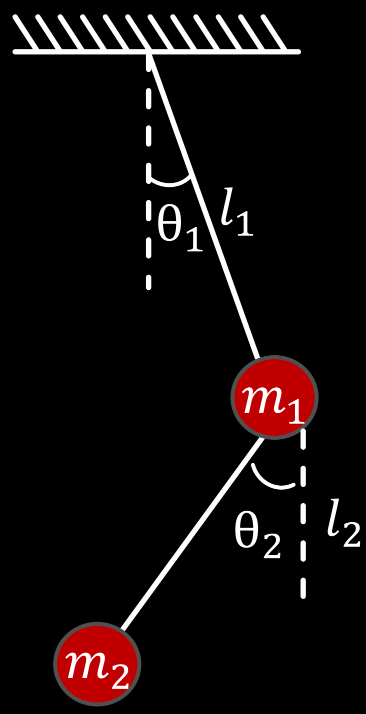
Newton's Equations



Derivatives

Kinematic Constraints

$$x_1 = l_1 \sin(\theta_1)$$

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$$

$$y_1 = -l_1 \sin(\theta_1)$$

$$y_2 = -l_1 \sin(\theta_1) - l_2 \sin(\theta_2)$$

Velocities

$$\dot{x}_1 = \dot{\theta}_1 l_1 \cos(\theta_1)$$

$$\dot{x}_2 = \dot{\theta}_1 l_1 \cos(\theta_1) + \dot{\theta}_2 l_2 \cos(\theta_2)$$

$$\dot{y}_1 = \dot{\theta}_1 l_1 \sin(\theta_1)$$

$$\dot{y}_2 = \dot{\theta}_1 l_1 \sin(\theta_1) + \dot{\theta}_2 l_2 \sin(\theta_2)$$

The acceleration is the second derivative.

$$\ddot{x}_1 = -\dot{\theta}_1^2 l_1 \sin(\theta_1) + \ddot{\theta}_1 l_1 \cos(\theta_1)$$

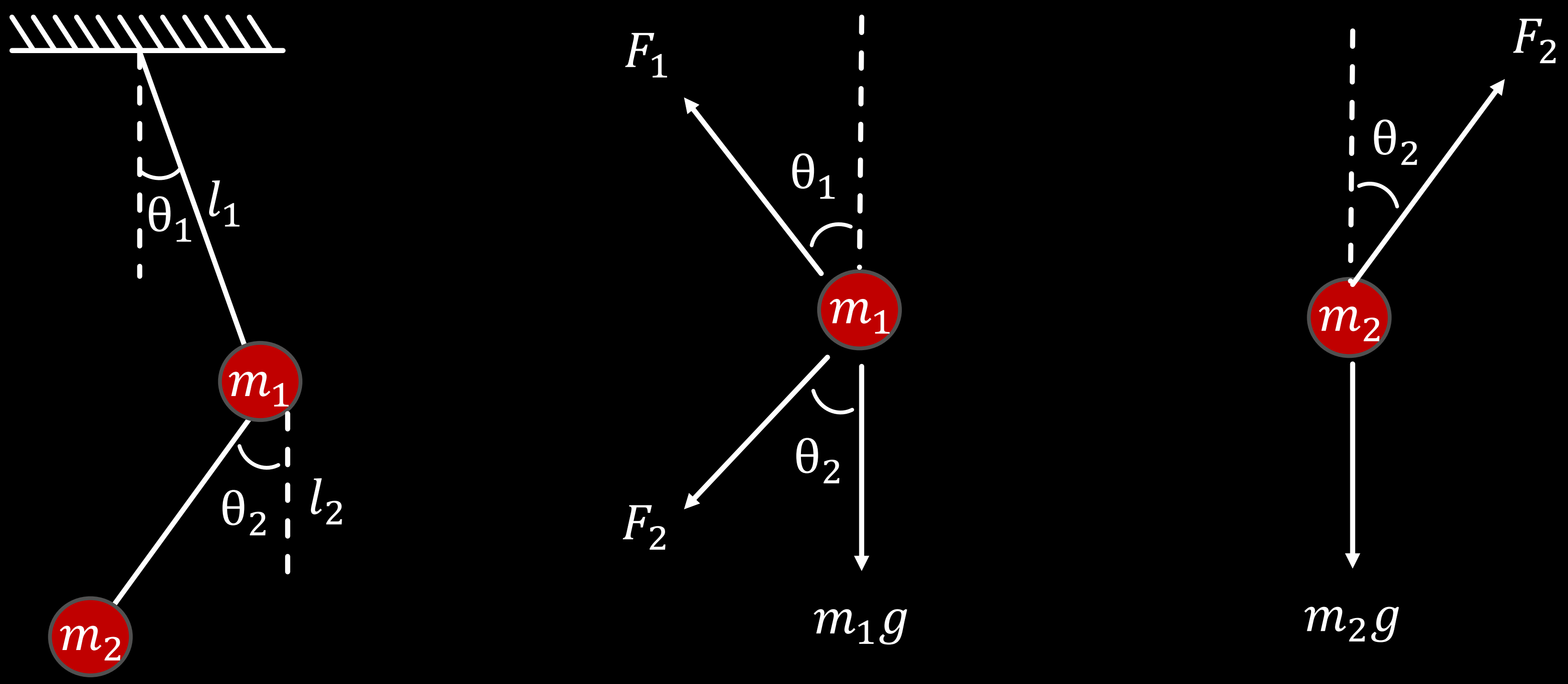
$$\ddot{y}_1 = \dot{\theta}_1^2 l_1 \cos(\theta_1) + \ddot{\theta}_1 l_1 \sin(\theta_1)$$

$$\ddot{x}_2 = -\dot{\theta}_1^2 l_1 \sin(\theta_1) + \ddot{\theta}_1 l_1 \cos(\theta_1) - \dot{\theta}_2^2 l_2 \sin(\theta_2) + \ddot{\theta}_2 l_2 \cos(\theta_2)$$

$$\ddot{y}_2 = -\dot{\theta}_1^2 l_1 \cos(\theta_1) + \ddot{\theta}_1 l_1 \sin(\theta_1) + \dot{\theta}_2^2 l_2 \cos(\theta_2) + \ddot{\theta}_2 l_2 \sin(\theta_2)$$

Newton's Equations

Forces in the Double Pendulum



Newton's Law Equation of Motion
m_1

$$\sum F = 0$$

$$m_1 \ddot{x}_1 = -F_1 sin\theta_1 + F_2 sin\theta_2$$

$$m_1 \ddot{y}_1 = F_1 cos\theta_1 - F_2 cos\theta_2 - m_1 g$$

Newton's Law Equation of Motion
m_2

$$m_2 \ddot{x}_2 = -F_2 sin\theta_2$$

$$m_2 \ddot{y}_2 = F_2 cos\theta_2 - m_2 g$$

Direct Method for Finding Equations of Motion:

- *Let's write the values of $m_2\ddot{x}_2$ and $m_2\ddot{y}_2$ in the equations of motion of mass m_1:*

$$m_1\ddot{x}_1 = -F_1 sin\theta_1 - m_2\ddot{x}_2$$

$$m_1\ddot{y}_1 = F_1 cos\theta_1 - m_2\ddot{y}_2 - m_2 g - m_1 g$$

- *Let's multiply the first equation by $cos\theta_1$ and the second equation by $sin\theta_1$:*

$$F_1 sin\theta_1 cos\theta_1 = -cos\theta_1(m_1\ddot{x}_1 + m_2\ddot{x}_2)$$

$$F_1 sin\theta_1 cos\theta_1 = sin\theta_1(m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_2 g + m_1 g)$$

- *If we equate the two equations:*

$$sin\theta_1(m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_2 g + m_1 g) = -cos\theta_1(m_1\ddot{x}_1 + m_2\ddot{x}_2)$$

- Then in the equation of motion for m_2, let's multiply $m_2\ddot{x}_2$ by $cos\theta_2$ and $m_2\ddot{y}_2$ by $sin\theta_2$.

$$F_2 sin\theta_2 cos\theta_2 = cos\theta_2(m_2\ddot{x}_2)$$

$$F_2 sin\theta_2 cos\theta_2 = sin\theta_2(m_2\ddot{y}_2 + m_2 g)$$

- If we equate the two equations:

$$sin\theta_2(m_2\ddot{y}_2 + m_2 g) = cos\theta_2(m_2\ddot{x}_2)$$

- Let's write the last equations we get:

$$sin\theta_1(m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_2 g + m_1 g) = -cos\theta_1(m_1\ddot{x}_1 + m_2\ddot{x}_2)$$

$$sin\theta_2(m_2\ddot{y}_2 + m_2 g) = cos\theta_2(m_2\ddot{x}_2)$$

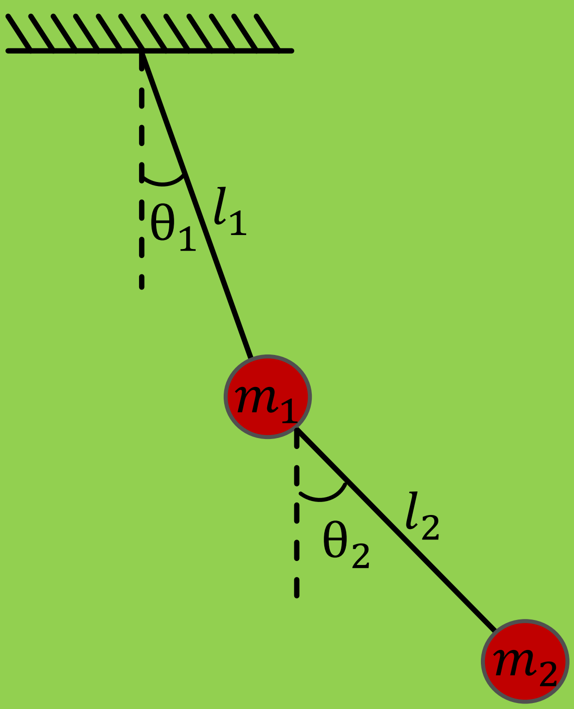
Newton's Law Equation of Motion

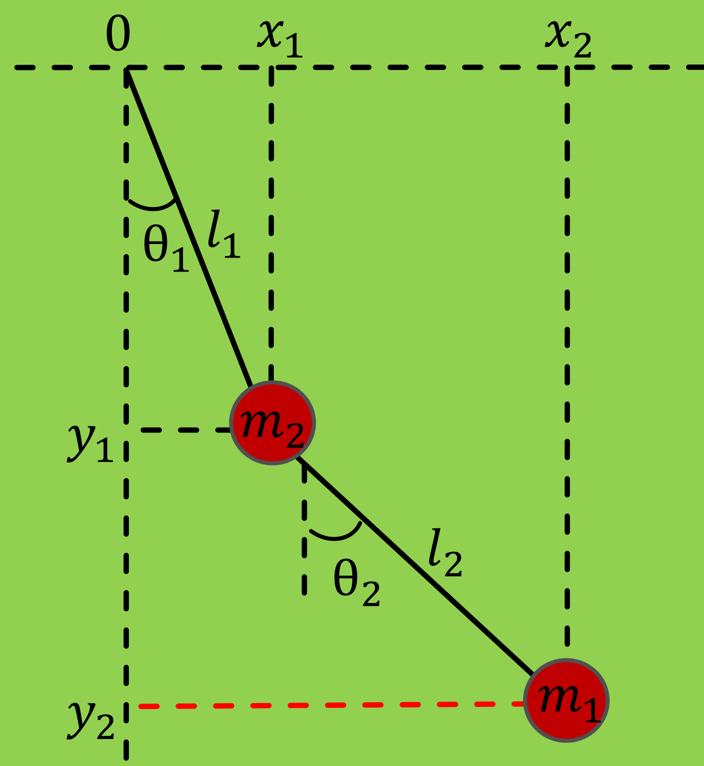
- *If we substitute the values of $\ddot{x}_1$, $\ddot{y}_1$, $\ddot{x}_2$ and $\ddot{y}_2$ into the equation and perform some complex calculations, the equations of motion for the system are obtained :*

$$\ddot{\theta}_1 = \frac{-g(m_1+m_2)sin\theta_1 - m_2 g sin(\theta_1 - 2\theta_2) - 2sin(\theta_1-\theta_2)m_2\left(\dot{\theta}_2^2 l_2 + \dot{\theta}_1^2 l_1 cos(\theta_1-\theta_2)\right)}{l_1\left(2m_1+m_2-m_2 cos(2\theta_1-2\theta_2)\right)}$$

$$\ddot{\theta}_1 = \frac{-2sin(\theta_1-\theta_2)\left(\dot{\theta}_1^2 l_1(m_1+m_2) + g(m_1+m_2)cos\theta_1 + l_2 m_2 cos(\theta_1-\theta_2)\right)}{l_2\left(2m_1+m_2-m_2 cos(2\theta_1-2\theta_2)\right)}$$

Lagrange's Equations





Derivatives

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Velocities

$$\dot{x}_1 = \dot{\theta}_1 l_1 \cos(\theta_1)$$

$$\dot{x}_2 = \dot{\theta}_1 l_1 \cos(\theta_1) + \dot{\theta}_2 l_2 \cos(\theta_2)$$

$$\dot{y}_1 = \dot{\theta}_1 l_1 \sin(\theta_1)$$

$$\dot{y}_2 = \dot{\theta}_1 l_1 \sin(\theta_1) + \dot{\theta}_2 l_2 \sin(\theta_2)$$

Lagrange Equations Steps:

- *Kinetic Energy of the System*
- *Potential Energy of the System*
- *L=K-P Lagrange's Equations*
- *Euler-Lagrange differential equation*

Kinetic Energy
Double Pendulum System

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$K = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$K = \frac{1}{2}m_1\left(l_1^2\dot{\theta}_1^2\overset{1}{\cancel{(cos^2\theta_1 + sin^2\theta_1)}}\right) + \frac{1}{2}m_2\Big[_1\left(\left(l_1\dot{\theta}_1 cos\theta_1 + l_2\dot{\theta}_2 cos\theta_2\right)^2\right)\Big]$$

$$+\left(l_1\dot{\theta}_1 sin\theta_1 + l_2\dot{\theta}_2 sin\theta_2\right)$$

$$K = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2\overset{1}{\cancel{(cos^2\theta_1 + sin^2\theta_1)}} + l_2^2\dot{\theta}_2^2(cos^2\theta_2 + sin^2\theta_2)]$$

$$+2l_1 l_2\theta_1\dot{\theta}_2(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2)$$

$$K = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\theta_1\dot{\theta}_2 cos(\theta_1 - \theta_2)\right)$$

Potential Energy
Double Pendulum System

$$P = m_1 g y_1 + m_2 g y_2$$

$$P = -(m_1 + m_2)g l_1 cos\theta_1 - m_2 g l_2 cos\theta_2$$

Lagrangian Equation
General

The Lagrangian(L) of a system is defined to be the difference of the kinetic energy and the potential energy.

$$K = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \theta_1 \dot{\theta}_2 cos(\theta_1 - \theta_2) \right)$$

$$P = -(m_1 + m_2) g l_1 cos\theta_1 - m_2 g l_2 cos\theta_2$$

$$L = K - P$$

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \theta_1 \dot{\theta}_2 cos(\theta_1 - \theta_2) \right)$$
$$+ (m_1 + m_2) g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2$$

Euler - Lagrangian Equation
Double Pendulum
$\boldsymbol{\theta_1}$

Euler-Lagrange differential equation : $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Partials of the Lagrangian for $\boldsymbol{\theta_1}$:

- $\frac{\partial L}{\partial \theta_1} = -l_1 g (m_1 + m_2) sin(\theta_1) - m_2 l_1 l_2 \theta_1 \theta_2 sin(\theta_1 - \theta_2)$

- $\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_1 - \theta_2)$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$

Lagrangian Equation
Double Pendulum
$\boldsymbol{\theta_1}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

- $(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) + l_1 g(m_1 + m_2) sin(\theta_1) = 0$

Dividing through by l_1 , this simplifies to :

- $(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) + l_1 g(m_1 + m_2) sin(\theta_1) = 0$

- $(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + g(m_1 + m_2) sin(\theta_1) = 0$

Simplifying and Solving for $\ddot{\theta}_1$:

$$\ddot{\theta}_1 = \frac{-m_2 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) - g(m_1 + m_2) sin(\theta_1)}{(m_1 + m_2) l_1}$$

Lagrangian Equation
General

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \theta_1 \dot{\theta}_2 cos(\theta_1 - \theta_2) \right)$$

$$+(m_1 + m_2) g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2$$

Euler - Lagrangian Equation
Double Pendulum
$\boldsymbol{\theta_2}$

Euler-Lagrange differential equation : $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Partials of the Lagrangian for $\boldsymbol{\theta_2}$:

- $\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) - l_2 m_2 g sin(\theta_2)$

- $\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 cos(\theta_1 - \theta_2)$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)$

Lagrangian Equation
Double Pendulum
θ_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

➢ $m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 sin(\theta_1 - \theta_2)$

$+m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) + m_2 g l_2 sin(\theta_2) = 0$

Dividing through by l_2 , this simplifies to :

➢ $m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 sin(\theta_1 - \theta_2)$

~~$+m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2)$~~ ~~$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2)$~~ $+ m_2 g l_2 sin(\theta_2) = 0$

➢ $m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 sin(\theta_1 - \theta_2) + m_2 g sin(\theta_2) = 0$

Simplifying and Solving for $\ddot{\theta}_1$:

$$\boxed{\ddot{\theta}_2 = \frac{-m_2 l_1 \ddot{\theta}_1 cos(\theta_1 - \theta_2) + m_2 l_1 \dot{\theta}_1^2 sin(\theta_1 - \theta_2) - m_2 g sin(\theta_2)}{m_2 l_2}}$$

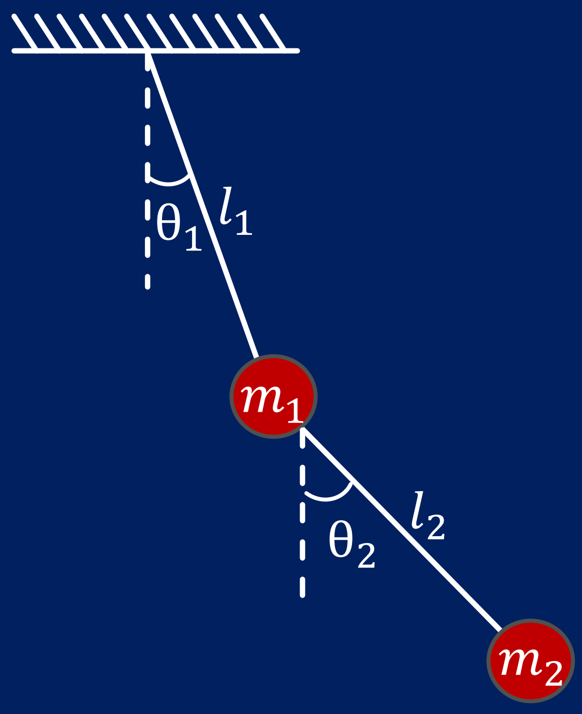
Euler - Lagrangian Equation

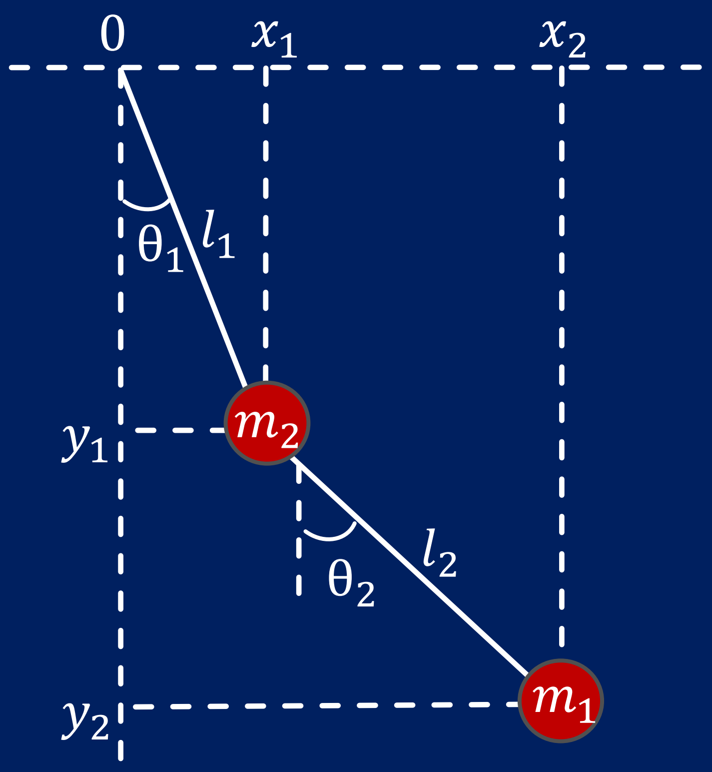
Double Pendulum Equation of Motion

$$\ddot{\theta}_1 = \frac{-m_2 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) - g(m_1 + m_2) sin(\theta_1)}{(m_1 + m_2) l_1}$$

$$\ddot{\theta}_2 = \frac{-m_2 l_1 \ddot{\theta}_1 cos(\theta_1 - \theta_2) + m_2 l_1 \dot{\theta}_1^2 sin(\theta_1 - \theta_2) - m_2 g sin(\theta_2)}{m_2 l_2}$$

Hamilton's Equations





- *Lagrangian Equation for Double Pendulum:*

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \theta_1 \dot{\theta}_2 cos(\theta_1 - \theta_2) \right) + (m_1 + m_2) g l_1 cos\theta_1$$
$$+ m_2 g l_2 cos\theta_2$$

- *The Hamiltonian of the System:*

$$H = \sum_{i=1}^{2} \dot{\theta}_i \, p_{\theta_i} - L \, ,$$

$$\dot{\theta}_i = \frac{\partial H}{\partial p_{\theta_i}} \, ,$$

$$p_{\theta_i} = - \frac{\partial H}{\partial \theta_i} \, , \quad \text{for} \quad i = 1,2$$

Hamilton's Equations

- *From the Lagrangian, one can obtain the canonical momenta of the system:*

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_1 - \theta_2)$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 cos(\theta_1 - \theta_2)$$

- *This can be written in the matrix form as:*

$$\begin{pmatrix} p_{\theta_1} \\ p_{\theta_2} \end{pmatrix} = B \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}, \quad B = \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_1 - \theta_2) \\ m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_1 - \theta_2) & m_2 l_2^2 \end{pmatrix}$$

- *The inverse transformation matrix form as:*

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = B^{-1} \begin{pmatrix} p_{\theta_1} \\ p_{\theta_2} \end{pmatrix}$$

$$\begin{aligned} \det(B) &= m_1 m_2 l_1^2 l_2^2 + m_2^2 l_1^2 l_2^2 [1 - cos^2(\theta_1 - \theta_2)] \\ &= m_1 m_2 l_1^2 l_2^2 + m_2^2 l_1^2 l_2^2 sin^2(\theta_1 - \theta_2) \\ &\geq m_1 m_2 l_1^2 l_2^2 \end{aligned}$$

- *B can be inverted directly, (2×2 matrix):*

$$B^{-1} = \frac{1}{det} \begin{pmatrix} m_2 l_2^2 & -m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_1 - \theta_2) \\ -m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_1 - \theta_2) & (m_1 + m_2) l_1^2 \end{pmatrix}$$

Hamilton's Equations

➢ After rearranging some terms:

$$\dot{\theta}_1 = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} cos(\theta_1 - \theta_2)}{l_1^2 l_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$$

$$\dot{\theta}_2 = \frac{-m_2 l_2 p_{\theta_1} cos(\theta_1 - \theta_2) - (m_1 + m_2) l_1 p_{\theta_2}}{m_2 l_1 l_2^2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$$

➢ The Hamiltonian $H = \sum_{i=1}^{2} \dot{\theta}_i \, p_{\theta_i} - L$ in terms of θ_1, θ_2, p_{θ_1} and p_{θ_2}:

$$H = \frac{-m_2^2 l_1^2 p_{\theta_1}^2 + (m_1 + m_2) l_1^2 p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1} p_{\theta_2} cos(\theta_1 - \theta_2)}{2 m_2 l_1^2 l_2^2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$$

$$-(m_1 + m_2) g l_1 cos\theta_1 - m_2 g l_2 cos\theta_2$$

➢ *Equation **H** can finally be used on equation* $\dot{\theta}_i = \frac{\partial H}{\partial p_{\theta_i}}$, $p_{\theta_i} = -\frac{\partial H}{\partial \theta_i}$ *to give us the Hamiltonian equations of motion for the double pendulum:*

$$\dot{\theta}_1 = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} cos(\theta_1 - \theta_2)}{l_1^2 l_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$$

$$\dot{\theta}_2 = \frac{-m_2 l_2 p_{\theta_1} cos(\theta_1 - \theta_2) - (m_1 + m_2) l_1 p_{\theta_2}}{m_2 l_1 l_2^2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$$

$$\dot{p}_{\theta_1} = \frac{\partial H}{\partial \dot{\theta}_1} = -(m_1 + m_2) g l_1 sin\theta_1 - h_1 + h_2 sin[2(\theta_1 - \theta_2)]$$

$$\dot{p}_{\theta_2} = \frac{\partial H}{\partial \dot{\theta}_2} = -m_2 g l_2 sin\theta_2 + h_1 - h_2 sin[2(\theta_1 - \theta_2)]$$

Hamilton's Equations

$$h_1 = \frac{p_{\theta_1} p_{\theta_2} sin(\theta_1 - \theta_2)}{l_1 l_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$$

$$h_1 = \frac{m_2 l_2^2 p_{\theta_1}^2 + (m_1 + m_2) l_1^2 p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1} p_{\theta_2} cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]^2}$$