Final Exam

Dynamic Optimization

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Open Book, Notes, Homework, Internet

Time Limit: 6 hours

Time(s) started: _____

Time(s) stopped: _____

Cumulative time:_____

Name_____

Exam Score

1. _____/30

2. _____/30

3. _____/40

Total:_____

1. (30 pts) Dynamic Optimization with Orthogonal Collocation on Finite Elements

Objective: Optimize an objective function subject to a differential equation. Use orthogonal collocation on finite elements to discretize the differential equation and the conditions of optimality to minimize the objective function.

Solve the following coupled differential equation from time starting at 0 until a final time of 1. Solve the system of equations with orthogonal collocation on finite elements with 3 nodes (time points) at $t = \begin{bmatrix} 0 & 0.5 & 1.0 \end{bmatrix}$ for discretization points.

$$\min_{u} \ (x - 3)^2$$

$$5\frac{dx}{dt} = -x + 2\,u$$
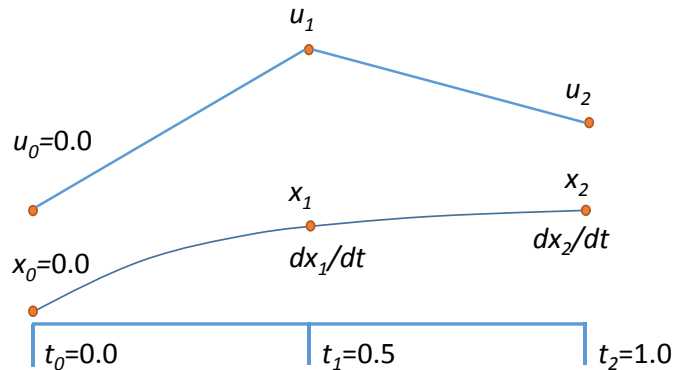
The initial condition, x_0, is 0 and the initial value of the manipulated variable, u_0, is also 0 (however, this initial condition doesn't affect the solution). The following is available for approximation of the derivative values at x_i, where i is the collocation point t_0, t_1, or t_2 (see link for additional details on orthogonal collocation):

$$t_2 \begin{bmatrix} 0.75 & -0.25 \\ 1.00 & 0.00 \end{bmatrix} \begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}$$

Differentiating the objective function with respect to x_1 and x_2 and setting those expressions equal to zero gives the final two equations necessary to satisfy the conditions of optimality (Karush-Kuhn-Tucker conditions).

Report the solution of u_1, u_2, x_1, and x_2 and derivative values ($\dot{x}_1$ and $\dot{x}_2$) at $t_1 = 0.5$ and $t_2 = 1.0$ (6 values total). Show work such as the system of 6 variables $\left(u_1, u_2, x_1, x_2, \dfrac{dx_1}{dt}, \dfrac{dx_2}{dt}\right)$ and 6 equations.

Hint: Because the problem is an unconstrained quadratic programming problem (quadratic objective, linear equations) the problem can be set up as a system of linear equations that are solved with a single matrix inversion. You can check your solution for (u_2, x_2) with the following, but show work for full credit:

| APM Model File (model.apm)	APM Data File (model.csv)	MATLAB Script
```Variables```	Time	```clear all; close all; clc```
``` u = 0```	0	```addpath('apm')```
``` x = 0```	1	```y = apm_solve('model',6);```
```Equations```		```disp(y.x.u); disp(y.x.u);```
``` minimize (x-3)^2```		
``` 5 * $x = -x + 2*u```		

2. (30 pts) Dynamic Optimization for Trajectory Planning

Objective: Design a model predictive controller for an inverted pendulum system with an adjustable cart. Demonstrate that the cart can perform a sequence of moves to maneuver from position y=-1.0 to y=0.0 within 6.2 seconds. Verify that v, θ, and q are zero before and after the maneuver.

The inverted pendulum is described by the following dynamic equations:

$$\begin{bmatrix} \dot{y} \\ \dot{v} \\ \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \\ \theta \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u$$

where u is the force applied to the cart, ϵ is $m_2/(m_1+m_2)$, y is the position of the cart, v is the velocity of the cart, θ is the angle of the pendulum relative to the cart, m_1=10, m_2=1, and q is the rate of angle change[1]. Tune the controller to minimize the use of force applied to the cart either in the forward or reverse direction (i.e. minimize fuel consumed to perform the maneuver). Explain the tuning and the optimal solution with appropriate plots that demonstrate that the solution is optimal. A similar but non-inverted pendulum problem is posted on the course web-site as an example in the introductory section for model predictive control.
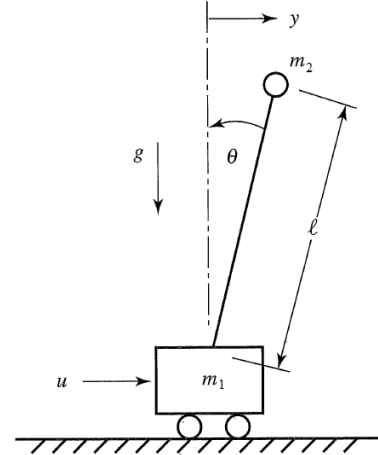
References

1. Bryson, A.E., Dynamic Optimization, Addison-Wesley, 1999.

3. (40 pts) Model Predictive Control of a Second Order Nonlinear System

Objective: Develop a model predictive controller (MPC) to reach a level set point for tank 2 (e.g. 50% full, level=0.5), initially starting from empty tanks. Tune the controller for satisfactory performance with setpoint changes in the level or with disturbances to the valve position (γ_1). Demonstrate the control performance and discuss methods used to achieve satisfactory control during startup from empty and with valve position changes (show both of these).

A pump transports water to tank 1 where it drains to tank 2. Both tanks have an opening at the bottom that allows liquid to flow out. A valve is used to divert pumped liquid into tank 1 (valve position=0), into tank 2 (valve position=1) or fractionally into both (valve position 0-1). The valve position is not available to the controller as a measurement and is considered a disturbance to the process. Only the level of tank 2 (bottom tank) is available as a measured value. The pump rate can be manipulated between 0 and 1. The pump has a maximum rate of change of 0.1 every second.

Note: For control you can use linear or nonlinear model predictive control. For estimation, you can use moving horizon estimation (MHE) or simple bias updating. The simplest option is linear MPC with bias updating. Below are some comments for implementing linear MPC for a Simulink implementation.
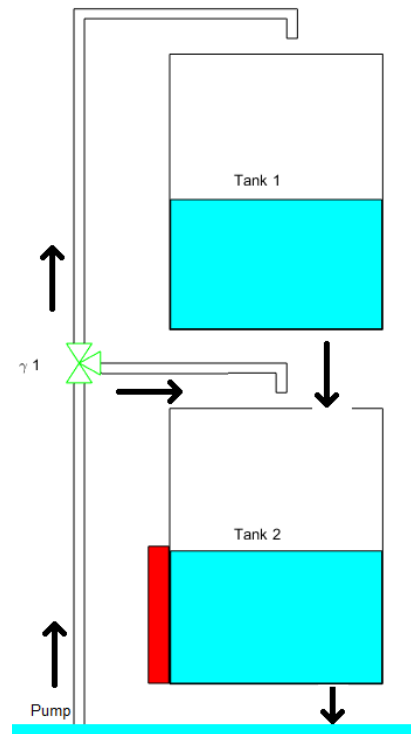
Conduct a step test by manipulating the pump rate and recording the level in the bottom tank. Use graphical approximation methods or Excel optimizer to obtain a FOPDT model with values of the process gain and time constant (Kp, τp). Implement linear MPC in the Simulink file by replacing the Pump Input block with a Triggered Subsystem for the controller. An example of linear MPC in Simulink is posted to the course web-site as a CSTR example. The files **mpc.apm**, **mpc.csv**, **mpc.m**, and **mpc_init.m** are located at this link and can be copied and modified for this application.
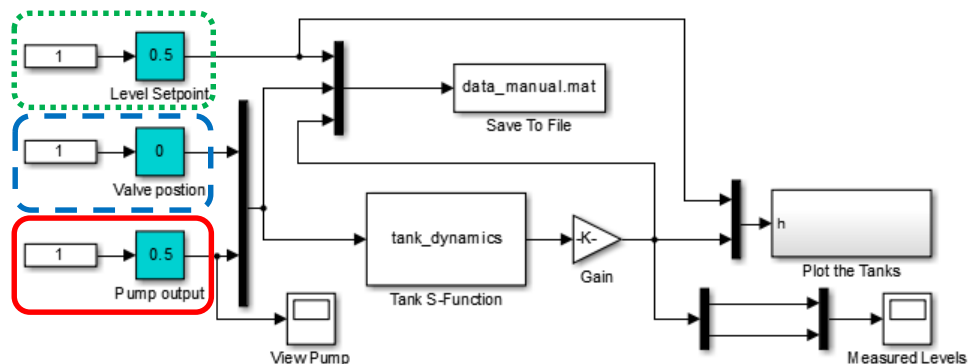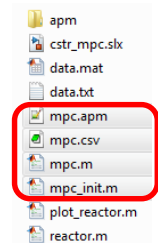
Figure 1. Design the MPC to manipulate the pump output by replacing the manual control (red). Demonstrate control performance with changes to the setpoint (green dotted) and valve position (blue dashed).

This following step test shows a gain (K_p) of approximately 1.1 (liquid level/pump output) and a time constant (τ_p) of approximately 100 s. This model is then $\tau_p\, d\,Level\,/\,dt = -(Level-Level_0) + K_p\,(Pump-Pump_0)$

with initial starting values *Level₀* and *Pump₀* at 0.0. This form of the model is not required for this application but gives a good starting point for MPC design.
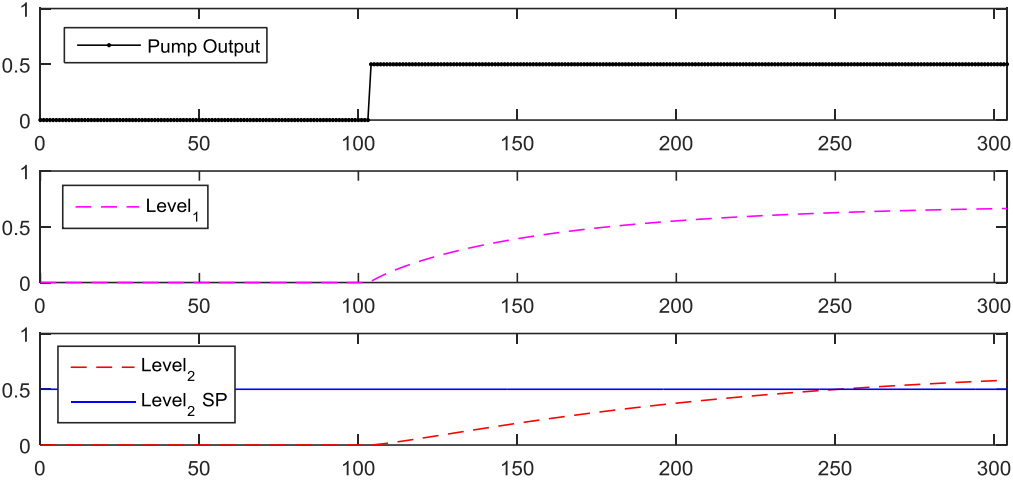

Figure 2. Step change in the pump rate from 0 to 0.5.